Weighted Quadrature Domains

and the Faber Transform

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PhD Examination

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Thesis Outline.

This thesis develops the theory of weighted quadrature domains in parallel with the Faber transform as a tool for proving various existence and uniqueness results.

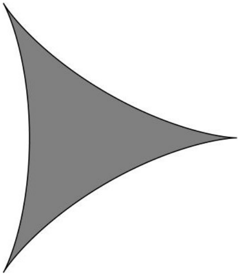
1. Classical quadrature domains (Chapter III)[1]
2. Power-Weighted quadrature domains (Chapter IV)[1]
3. Log-Weighted quadrature domains (Chapter V)[1,2]
4. Algebraic quadrature domains. (Chapter VI)

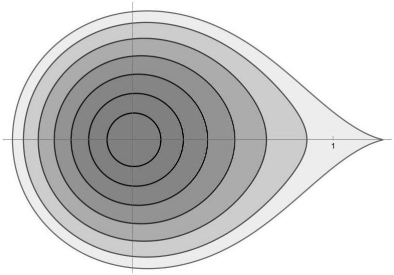
[1] Andrew Graven and Nikolai G. Makarov (Sept. 2025). Quadrature Domains and the Faber Transform. arXiv: 2509.03777[math.CV]. url: https://arxiv.org/abs/2509.03777.

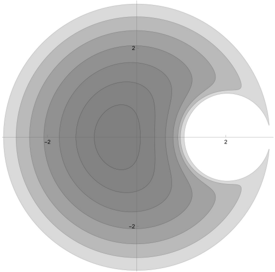
[2] Andrew Graven (Apr. 2026). Analysis of Log-Weighted Quadrature Domains. arXiv: 2604.10394 [math.CV]. url: https://arxiv.org/abs/2604.10394.

Chapter III: Classical Quadrature Domains

(Graven-Makarov, 2025)[1]







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Background (early examples of quadrature domains)

Mean value property of the disk:

$$f \in \mathcal{A}(\mathbb{D}_r(w_0)) \implies \frac{1}{r^2}\int_{\mathbb{D}_r(w_0)} f dA = f(w_0).$$

Epstein & Schiffer (1965): $\mathbb{D}_r(w_0)$ is the only[3] domain with this property.

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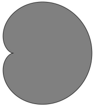

$$f \in \mathcal{A}(\Omega) \implies \int_\Omega f dA = \frac{3}{2}f(0) + \frac{1}{2}f'(0).$$

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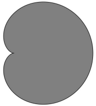

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Aharonov & Shapiro (1976): The cardioid is also unique.

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Bounded Quadrature Domains

Definition 1.1 (Bounded Quadrature domain)

A bounded domain $\Omega \subset \widehat{\mathbb{C}}$ is a *bounded* QD if there exists rational h s.t.

$$\int_{\Omega} f dA = \oint_{\partial\Omega} f(w)h(w)dw$$

$\forall f \in \mathcal{A}(\Omega)$. This is denoted by $\Omega \in \mathrm{QD}(h)$.

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This is equivalent to the existence of a quadrature identity:

$$\oint_{\partial\Omega} f(w)h(w)dw = \sum_{p_k \in h^{-1}(\infty)} \operatorname*{Res}_{w=p_k} (f(w)h(w)) = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k).$$

Unbounded Quadrature Domains

Definition 1.2 (Unbounded Quadrature Domain)

An unbounded domain $\Omega \subset \widehat{\mathbb{C}}$ is an *unbounded* QD if $\exists$ a rational h s.t.

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unbounded QD $\longleftrightarrow$ quadrature identity

$$\oint_{\partial\Omega} f(w)h(w)dw = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k) + \sum_j c_j f_j$$

where $f(w) = \sum_{j=1}^{\infty} f_j w^{-j}$.

Characterizations of QDs I: Electrostatics

Definition 1.3 (Cauchy transform)

For a Borel set $\Omega \subset \mathbb{C}$, we denote the *Cauchy transform* of Ω by $C^\Omega : \mathbb{C} \to \mathbb{C}$,

$$C^\Omega(w) = \int_\Omega \frac{dA(\xi)}{w - \xi}$$

$$\Omega \in \mathrm{QD}(h) \iff C^\Omega = h \text{ in } \Omega_*$$

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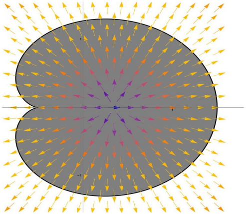
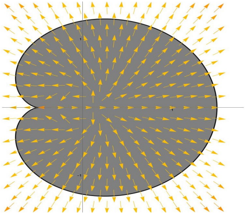
$\overline{C^\Omega}$ corresponds to the electric field due to a uniform charge distribution on Ω.



$\overline{C^\Omega}$



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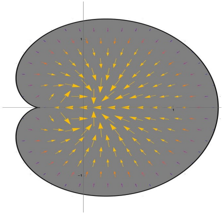

$\overline{C^\Omega} - \overline{h}$

Characterizations of QDs II: The Coincidence Equation

$\Omega \subset \widehat{\mathbb{C}}$ is a QD iff there exists a rational function h such that

$$\overline{w} = h(w) + C^{\Omega_*}(w), \quad \forall w \in \partial\Omega.$$

Referred to as the *coincidence equation*.

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Referred to as the *coincidence equation*.

We interpret this identity in terms of logarithmic potential theory:
If $K = \Omega^c$ is a local droplet of the potential

$$Q(w) = |w|^2 - 2\mathrm{Re}(H(w)),$$

then

$$\frac{\partial}{\partial \overline{w}} (Q(w) + U^{\mu}(w)) = 0, \quad \forall w \in K.$$

If we set $h = H'$ then differentiating, yields the coincidence equation.

Characterizations of QDs III: The Schwarz Function

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$$\Omega \subset \widehat{\mathbb{C}} \text{ is a QD} \iff \Omega \text{ admits a } \textit{Schwarz function}.$$

S is called a *Schwarz function* for Ω if:

1. $S \in \mathcal{M}(\Omega)$,
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S can be represented explicitly in terms of h and the Cauchy transform:

$$S(w) = h(w) + C^{\Omega_*}(w).$$

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If Ω is simply connected, then $\Omega \in QD$ iff its Riemann map is rational.

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Moreover, if $\Omega \in \mathrm{QD}(h)$ is a **bounded** domain with Riemann map φ, then

$$h(w) = \Phi_\varphi \left(\varphi^\# - \overline{\varphi(0)} \right)(w)$$

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If Ω is **unbounded**, then

$$h(w) = \Phi_\varphi \left(\varphi^\# - cz^{-1}\right)(w)$$

$$\varphi(z) = cz + \Phi_\varphi^{-1}(h)^\#(z)$$

Where Φ_φ is the *Faber transform*.

The Faber Transform

Let $\Omega \subset \widehat{\mathbb{C}}$ be unbounded and simply connected with Riemann map $\varphi : \mathbb{D}_* \to \Omega$,

$$\varphi(z) = cz + f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots, \qquad c = \mathrm{rad}_\infty(\Omega) > 0$$

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The associated *exterior Faber transform* Φ_φ is a linear iso $\mathcal{A}(\mathbb{D}) \to \mathcal{A}(\Omega_*)$,

$$\Phi_\varphi(f)(w) = \oint_{\partial\mathbb{D}_*} \frac{f(z)\varphi'(z)}{\varphi(z) - w} dz$$

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Note: the Faber transform preserves polynomials and rational functions, e.g.

$$\Phi_\varphi\left(\frac{1}{z - z_0}\right)(w) = \frac{\varphi'(z_0)}{w - \varphi(z_0)},$$

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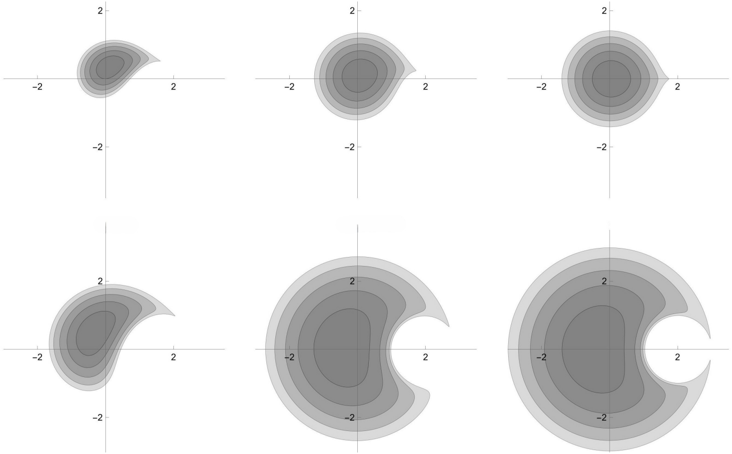
The *interior* Faber transform, defined for bounded s.c. domains is defined similarly, and is a map $\Phi_\varphi : \mathcal{A}_0(\mathbb{D}_*) \to \mathcal{A}_0(\Omega_*)$.

Classification of One-Point Quadrature Domains



Existence and Uniqueness of One Pt. UQDs

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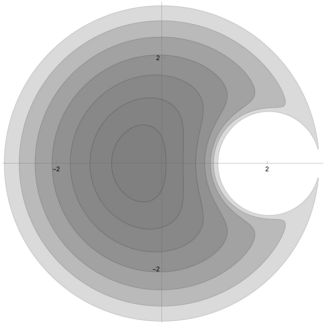

One point UQDs for $w_0 = 2$, $\alpha = 1$

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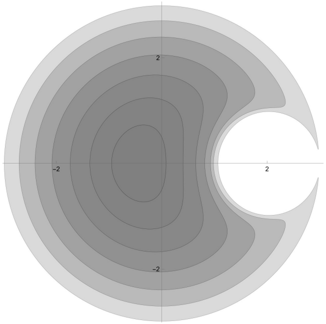

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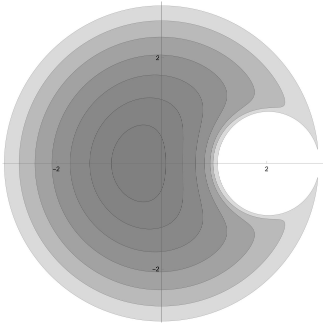
By change of variables,

$$\Omega \in \mathrm{QD}\left(\frac{\alpha}{w - w_0}\right) \iff \frac{2}{w_0}\Omega \in \mathrm{QD}\left(\frac{\frac{4\alpha}{|w_0|^2}}{w-2}\right)$$

so we can wlog assume $w_0 = 2$.



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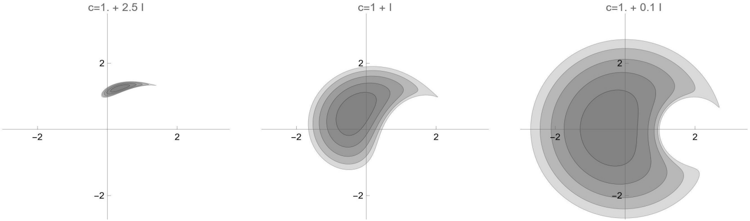
Existence and Uniqueness of One-Point UQDs

Theorem 2.1

Take $\alpha \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and $w_0 \neq 0$. Then

$$QD\left(\frac{\alpha}{w-w_0}\right) \neq \emptyset \qquad \iff \qquad |w_0|^2 + 2Re(\alpha) > 2|\alpha|.$$



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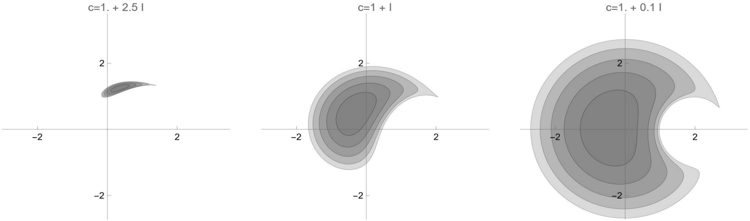
In this case, there exists $t_* \in (0, \infty)$ such that

1. *$QD\left(\frac{\alpha}{w-w_0}\right) = \{\Omega_t\}_{0<t\leq t_*}$, a continuous monotone family of simply connected domains (where $t = A(\Omega_t)$);*



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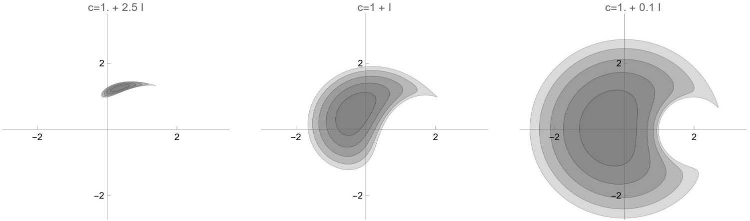
1. $QD\left(\frac{\alpha}{w-w_0}\right) = \{\Omega_t\}_{0<t\leq t_*}$, *a continuous monotone family of simply connected domains (where* $t = A(\Omega_t)$*);*
2. $\partial\Omega_t$ *is smooth for all* $0 < t < t_*$ *and the family terminates with the formation of a* $(3, 2)$ *cusp at* $t = t_*$.



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Outline of Proof

$$\Omega_t \in \mathrm{QD}(h), \qquad h(w) = \frac{\alpha}{w-2}$$

Key idea Use correspondence between unbounded QDs and local droplets of the associated potential:

$$Q(w) = |w|^2 - 2\mathrm{Re}\left(\alpha \ln(w-2)\right).$$

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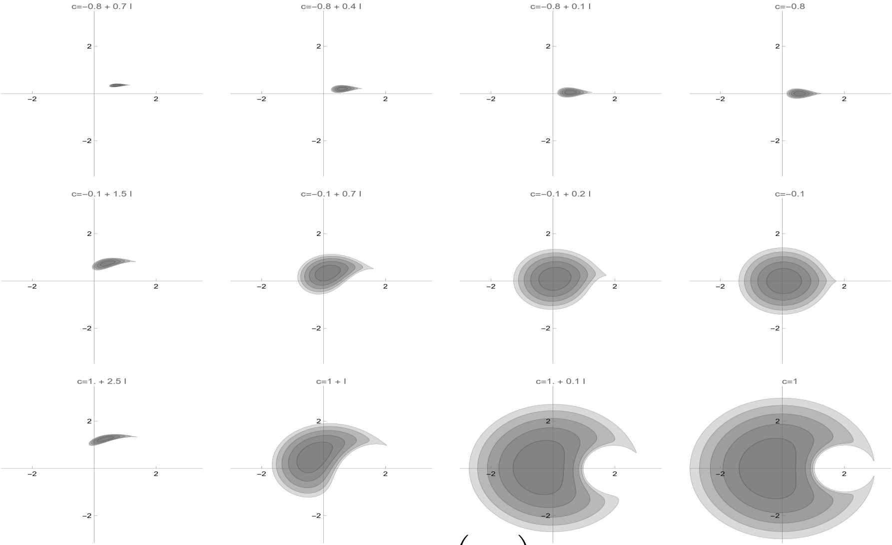
3. $(3,2)$ cusp development: Use Faber transform to obtain representation of Riemann map φ_t; show φ_t is not univalent for $t >> 0$; show a $(3,2)$ cusp must form during transition out of univalence.

Some Examples.



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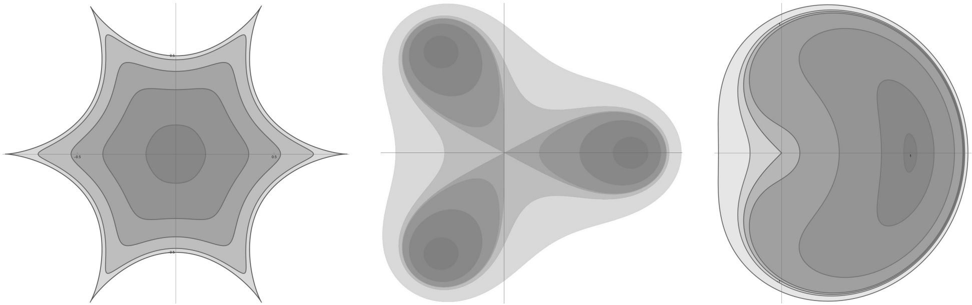
Chapter IV: Power-Weighted Quadrature Domains

(Graven-Makarov, 2025)[1]



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for all $f \in \mathcal{A}(\Omega)$. This is denoted by $\Omega \in \mathrm{QD}_a(h)$.

PQDs are weighted quadrature domains with respect to the weight:

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- The primary distinction of PQDs from classical QDs is the singular point of $\rho_a(w)$ at $w = 0$.
- The singular point at 0 results in boundary behavior not present in the classical case (formation of sharp corners at $w = 0$).

Some Basic Properties

Theorem 3.2 (Boundary regularity)

If $\Omega \in QD_a$ then $\partial\Omega$ has finitely many singular points, where each **non-zero** *singular point is either a cusp or a double point. $(a \in \mathbb{Z}_+)$*

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Theorem 3.3 (Algebraicity)

If $\Omega \in QD_a$ then $(\partial\Omega)^a$ is a subset of an algebraic curve. $(a \in \mathbb{Z}_+)$

Some Basic Properties

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If $\Omega \in QD_a$ then $\partial\Omega$ has finitely many singular points, where each **non-zero** *singular point is either a cusp or a double point. ($a \in \mathbb{Z}_+$)*

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For each $\alpha \in \mathbb{C} \setminus \{0\}$, $\Omega \in QD_a(h)$ if and only if $\alpha\Omega \in QD_a\left(\overline{\alpha}|\alpha|^{2(a-1)} h\left(\frac{w}{\alpha}\right)\right)$

Characterizations of PQDs I: The Coincidence Equation

Theorem 3.5

A domain $\Omega \subset \widehat{\mathbb{C}}$ is a PQD if and only if there exists $a > 0$, $G \in \mathcal{A}(\Omega)$, and a rational function h such that

$$\frac{1}{a}\overline{w}|w|^{2(a-1)} = h(w) + G(w), \quad \forall w \in \partial\Omega.$$

This is the *generalized coincidence equation*.

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This is the *generalized coincidence equation*.

We interpret this identity in terms of logarithmic potential theory:

If $K = \Omega^c$ is a local droplet of the potential

$$Q(w) = \frac{|w|^{2a}}{a^2} - 2\mathrm{Re}(H(w)),$$

then

$$\frac{\partial}{\partial \overline{w}}(Q(w) + U^{\mu}(w)) = 0, \quad \forall w \in K.$$

If we set $h = H'$ then differentiating, yields the coincidence equation.

Characterizations of PQDs II: The Gen. Schwarz Function

Theorem 3.6

$\Omega \subset \widehat{\mathbb{C}}$ is a PQD if and only if Ω admits a generalized Schwarz function S_a.

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S_a is a *generalized Schwarz function* for Ω if:

1. $S_a \in \mathcal{M}(\Omega)$,
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If $\Omega \in \mathrm{QD}_a(h)$, then S_a can be represented explicitly in terms of h and the weighted Cauchy transform:

$$S_a(w) = h(w) + C_{\rho_a}^{\Omega_*}(w).$$

Characterizations of PQDs III: The Riemann Map

Let Ω be a simply connected domain with Riemann map φ.

Write the *inner/outer factorization* of φ:

$$\varphi(z) = \varphi_{\text{in}}(z)\varphi_{\text{out}}(z).$$

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In general:

- φ_{in} is a ratio of Blaschke factors $b_{z_0}(z) := \frac{\overline{z_0}}{|z_0|}\frac{z-z_0}{z\overline{z_0}-1}$,
- φ_{out} is non-zero and analytic.

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- φ_{out} is non-zero and analytic.

This yields a method of solving the **inverse problem** for simply connected domains:

Given a rational function h, find a simply connected domain $\Omega \in QD_a(h)$.

Idea: Represent Ω via its Riemann map $\varphi = \varphi_{\mathrm{in}}\varphi_{\mathrm{out}}$ (consisting of rational functions) and solve for the coefficients.

The Faber Transform Method For Bounded PQDs

Theorem 3.8

Let $\Omega \in QD_a(h)$ be a bounded simply connected domain with Riemann map $\varphi : \mathbb{D} \to \Omega$. Then there exists rational $r \in \mathcal{A}_0(\mathbb{D}_*)$ such that:

$$\varphi(z) = r^{\#}(z)^{\frac{1}{a}}, \qquad \text{when } 0 \notin \Omega,$$

$$\varphi(z) = b_{z_0}(z) r^{\#}(z)^{\frac{1}{a}}, \quad \text{when } 0 \in \Omega.$$

Where $\varphi(z_0) = 0$.

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Where $\varphi(z_0) = 0$. In this case,

$$r = a\Phi_{\varphi}^{-1}\left(\mathcal{C}_{\Omega_*}\left[wh(w)\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right) + C,$$

and

$$h(w) = \frac{1}{aw}\Phi_{\varphi}\left(\mathcal{C}_{\mathbb{D}_*}\left[rr^{\#}\right]\right)(w) + \frac{t}{w},$$

where $t = \int_{\Omega} |w|^{2(a-1)} dA(w)$.

Theorem 3.9

Let $\Omega \in QD_a(h)$ be an unbounded simply connected domain with Riemann map $\varphi : \mathbb{D}_* \to \Omega$. Then there exists rational $r \in \mathcal{A}(\mathbb{D})$ such that:

$$\varphi(z) = zr^{\#}(z)^{\frac{1}{a}}, \qquad \text{when } 0 \notin \Omega,$$

$$\varphi(z) = zb_{z_0}(z)r^{\#}(z)^{\frac{1}{a}}, \quad \text{when } 0 \in \Omega.$$

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where $t = \int_{\Omega_*} |w|^{2(a-1)} dA(w)$.

One-Point PQDs.

Let $0 \notin \Omega$ be an unbounded simply connected domain. Then:

Theorem 3.10 (Characterization of non-singular unbounded one-point PQDs)

$\Omega \in QD_a\left(\frac{\alpha}{w-w_0}\right)$ if and only if $\Omega = \varphi(\mathbb{D}_*)$, where $\varphi : \mathbb{D}_* \to \widehat{\mathbb{C}}$ is a univalent map of the form

$$\varphi(z) = cz\left(1 + \frac{|z_0|^2 - 1}{z\overline{z_0} - 1}(\beta^a - 1)\right)^{\frac{1}{a}},$$

where $\beta = \frac{w_0}{cz_0}$ and $\varphi(z_0) = w_0$.

One-Point PQDs

Let $0 \in \Omega$ be an unbounded simply connected domain. Then:

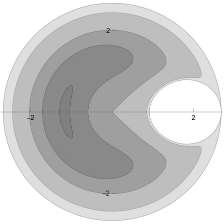
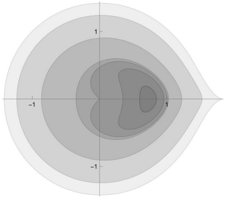
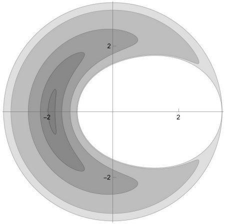
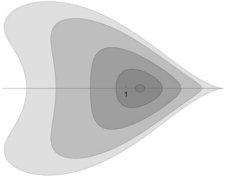
Theorem 3.11 (Characterization of singular unbounded one-point PQDs)

$\Omega \in QD_a\left(\frac{\alpha}{w-w_0}\right)$ if and only if $\Omega = \varphi(\mathbb{D}_*)$, where $\varphi : \mathbb{D}_* \to \widehat{\mathbb{C}}$ is a univalent map of the form

$$\varphi(z) = cz|z_1|b_{z_1}(z)\left(1 + \frac{|z_0|^2 - 1}{z\overline{z_0} - 1}(\beta^a - 1)\right)^{\frac{1}{a}},$$

where $\beta = \frac{w_0}{cz_0}\frac{z_0 - \overline{z_1}^{-1}}{z_0 - z_1}$, $\varphi(z_0) = w_0$, and $\varphi(z_1) = 0$.

Example: $\Omega \in \mathbf{QD}_2\left(\frac{\alpha}{w-2}\right)$.

	$\alpha > 0$	$\alpha < 0$
$0 \notin \Omega_{t_*}$		
$0 \in \Omega_{t_*}$		

Monomial PQDs.

Theorem 3.12 (Non-singular constant PQDs)

Take $a > 0$ and $\alpha \in \mathbb{C} \setminus \{0\}$. There exists a simply connected domain $0 \notin \Omega$, of conformal radius $c > 0$, for which $\Omega \in QD_a(\alpha)$ if and only if either

1. $0 < a < \frac{1}{2}$ *and* $|\gamma| \leq \frac{a}{1-a}$,
2. $a > \frac{1}{2}$ *and* $|\gamma| \leq 1$,
3. $a = \frac{1}{2}$ *and* $|\alpha| \leq 2$.

where $\gamma = -\frac{a\overline{\alpha}}{c^{2a-1}}$. *In this case,* Ω *is unique modulo conformal radius, and* $\Omega = \varphi(\mathbb{D}_*)$ *with* φ *univalent in* $\mathbb{D}_*$ *and given by:*

$$\varphi(z) = cz\left(1 - \frac{\gamma}{z}\right)^{\frac{1}{a}}.$$

Monomial PQDs.

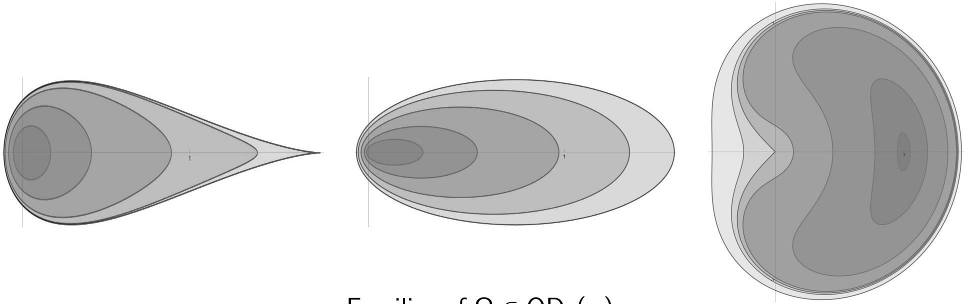
Theorem 3.13 (Singular constant PQDs)

Take $a > 0$ and $\alpha \in \mathbb{C} \setminus \{0\}$. There exists a simply connected domain Ω of conformal radius $c > 0$ containing 0 for which $\Omega \in QD_a(\alpha)$ if and only if $|\gamma| \geq 1$, with $\gamma = -\frac{a\overline{\alpha}}{c^{2a-1}}$. In this case, Ω is unique modulo conformal radius, and $\Omega = \varphi(\mathbb{D}_*)$ with φ given by

$$\varphi(z) = cz\frac{z - \overline{\gamma}^{\frac{1}{2a-1}}}{z - \gamma^{-\frac{1}{2a-1}}}\left(1 - \frac{\gamma^{\frac{1}{2a-1}}}{z}\right)^{\frac{1}{a}}.$$

Monomial PQDs.

$a < \frac{1}{2}$ (terminal) $\quad$ $a = \frac{1}{2}$ (traveling wave) $\quad$ $a > \frac{1}{2}$ (global)



Families of $\Omega \in QD_a(\alpha)$.

Monomial PQDs.

Theorem 3.14 (Non-singular symmetric monomial PQDs)

Fix $a > 0$, $\alpha \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{Z}_+$. If $0 \notin \Omega$ is a $\mathbb{Z}_k$−rotationally symmetric domain ($e^{\frac{2\pi i}{k}}\Omega = \Omega$), then $\Omega \in QD_a(\alpha k w^{k-1})$ if and only if $\Omega^k \in QD_{\frac{a}{k}}(\alpha k^2)$. In this case, $\Omega = \widetilde{\varphi}(\mathbb{D}_*)$, where

$$\widetilde{\varphi}(z) = cz\left(1 - \frac{\gamma_k}{z^k}\right)^{\frac{1}{a}}, \quad \gamma_k = -\frac{a\overline{\alpha}k}{c^{2a-k}}. \tag{1}$$

is univalent in $\mathbb{D}_*$.

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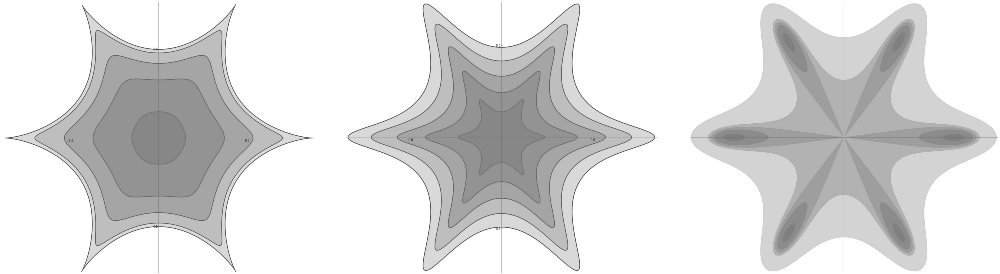
$a < \frac{k}{2}$ (terminal) $\qquad a = \frac{k}{2}$ (traveling wave) $\qquad a > \frac{k}{2}$ (global)



Families of $\Omega \in QD_a(\alpha k w^{k-1})$.

Quadratic PQDs.

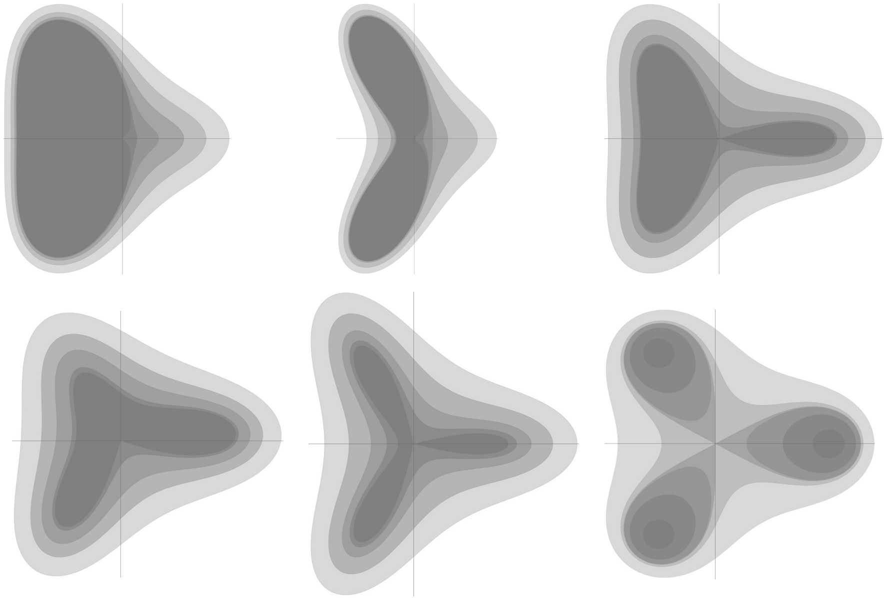
If $QD_2(\alpha_0 + \alpha_1 w + w^2)$ is non-singular and simply connected with Riemann map $\varphi : \mathbb{D}_* \to \Omega$, then

$$\varphi(z) = cz\left(1 + \frac{\overline{c_1}}{z} + \frac{2\overline{\alpha_1} + c_1 c}{c^2 z^2} + \frac{2}{cz^3}\right)^{\frac{1}{2}}$$

where c_1 is a root of the quartic:

$$\begin{aligned} 0 =& 64(\alpha_1 + \overline{\alpha_0})^2 - 128(c^2 - 1)^2(\overline{\alpha_1} + \alpha_0) \\ &+ 64c(c^2 - 1)^3 c_1 - 16c^2(\alpha_1 + \overline{\alpha_0})c_1^2 + c^4 c_1^4. \end{aligned}$$

Quadratic PQD Examples.



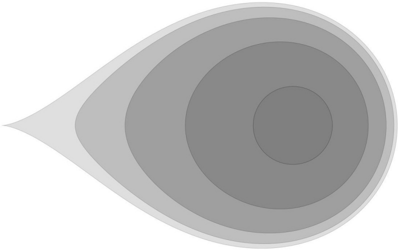
Non-uniqueness of PQDs

Quadrature domain uniqueness conjecture: *if $\Omega \in QD(h)$ is simply connected, then it is unique (modulo conformal radius).*

The following example demonstrates that the generalization of this conjecture to PQDs is false. We consider $\Omega = \varphi(\mathbb{D}_*)$, where:

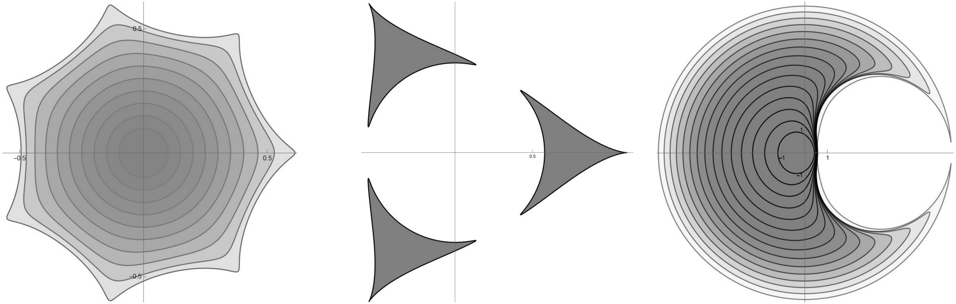
$$\varphi(z) = c\,(z+u)\sqrt{\frac{z - 2\gamma u^{-3}}{z + u^{-1}}}$$

where $\gamma = -\frac{2\alpha}{c^2}$, $u = \sqrt{\sqrt{\gamma\,(\gamma+4)} - \gamma}$, and $c, \alpha > 0$. Then $\pm\Omega \in \mathrm{QD}_2(2\alpha w)$.



Chapter V: Log-Weighted Quadrature Domains

(Graven, 2026)[2]



[2] Andrew Graven (Apr. 2026). Analysis of Log-Weighted Quadrature Domains. arXiv: 2604.10394 [math.CV]. url: https://arxiv.org/abs/2604.10394.

Log-Weighted Quadrature Domains

Definition 4.1 (Log-Weighted Quadrature Domain)

Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain for which $0, \infty \notin \partial\Omega$. We say that Ω is a log-weighted quadrature domain (LQD) if there exists a rational h such that

$$\int_{\Omega} f(w)|w|^{-2} dA(w) = \oint_{\partial\Omega} f(w)h(w)dw, \qquad \forall f \in L^1_a(\Omega; \rho_0).$$

Denoted by $\Omega \in \mathrm{QD}_0(h)$.

LQDs are weighted QDs with respect to the weight: $\rho_0(w) = |w|^{-2}$.

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LQDs are weighted QDs with respect to the weight: $\rho_0(w) = |w|^{-2}$.

- The primary distinction of LQDs from classical QDs is the non-integrable singularity of the weight at $w = 0$.
- The singularity plays a similar role to $\infty \in \widehat{\mathbb{C}}$ for classical QDs.
- h is *not* always unique: If $0 \in \Omega \in \mathrm{QD}_0(h)$, then $\Omega \in \mathrm{QD}_0\left(h(w) + \frac{q}{w}\right)$ for all $q \in \mathbb{C}$.

A First Example

Fact: If Ω is a bounded domain, then

$$\Omega \in \mathrm{QD}(h) \implies e^{\Omega} \in \mathrm{QD}_0(\widetilde{h}),$$

for some rational $\widetilde{h}$. Moreover, if $\{p_k\}$ are the poles of h then $\{e^{p_k}\}$ are the poles of $\widetilde{h}$, with the same multiplicity.

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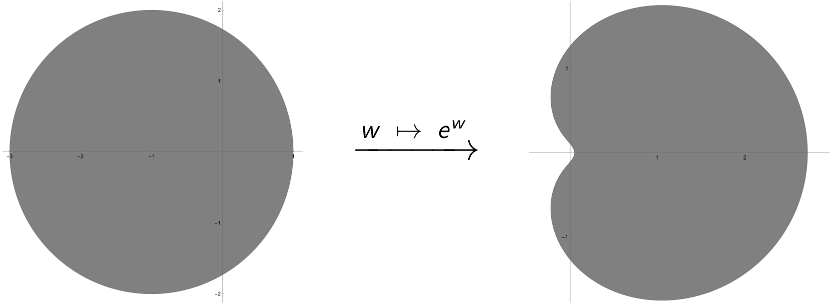
$$\mathbb{D}_2(-1) \in QD\left(\frac{4}{w+1}\right) \implies e^{\mathbb{D}_2(-1)} \in QD_0\left(\frac{4}{w-e^{-1}}\right)$$



Basic Properties

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Theorem 4.4 (Invariance under inversion)

If $\Omega \subseteq \widehat{\mathbb{C}}$ then $\Omega \in QD_0(h)$ iff $\Omega^{-1} \in QD_0(-h(w^{-1})w^{-2})$.

(the singularities at $w = 0$ and $w = \infty$ are "interchangable")

Characterizations of LQDs I: Electrostatics

Definition 4.5 (Weighted Cauchy transform)

For a Borel set $\Omega \subset \mathbb{C}$, we denote the ρ_0-*weighted Cauchy transform* of Ω by $C^{\Omega}_{\rho_0} : \mathbb{C} \to \mathbb{C}$,

$$C^{\Omega}_{\rho_0}(w) = \int_{\Omega} \frac{|\xi|^{-2} dA(\xi)}{w - \xi}$$

$$\Omega \in \mathrm{QD}_0 \iff C^{\Omega}_{\rho_0} \text{ extends to a rational function.}$$

$\overline{C^{\Omega}_{\rho_0}}$ corresponds to the electric field due to a charge density $\rho_0(w) = |w|^{-2}$ on Ω:

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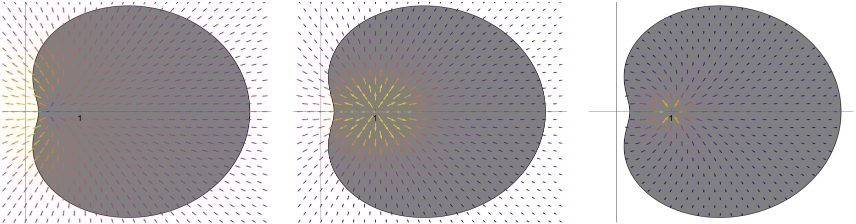

$\overline{C^{\Omega}_{\rho_0}}$ $\qquad$ $\overline{h}$ $\qquad$ $\overline{C^{\Omega}_{\rho_0}} - \overline{h}$

Characterizations of LQDs II: The Coincidence Equation.

Theorem 4.6

A domain Ω is an LQD if and only if there exist $q \in \mathbb{C}$ and $G \in \mathcal{A}(\Omega)$ such that

$$\frac{\ln|w|^2}{w} = h(w) + \frac{q}{w} + G(w), \quad \text{on} \ \ \partial\Omega.$$

In this case, $\Omega \in QD_0(h)$.

Characterizations of LQDs III: The Generalized S-Function

Theorem 4.7

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Remark: Solving for $\overline{w}$, we recover an analogue of the classical Schwarz function $S|_{\partial\Omega}(w) = \overline{w}$:

$$S(w) := \frac{1}{w} e^{wS_0(w)}.$$

Not typically a Schwarz function (essential singularity $\implies$ not meromorphic).

Characterizations of LQDs IV: The Riemann Map

Let Ω be a simply connected domain with Riemann map φ.

Write the *inner/outer factorization* of φ:

$$\varphi(z) = \varphi_{\mathrm{in}}(z)\varphi_{\mathrm{out}}(z).$$

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In general:

- φ_{in} is a ratio of Blaschke factors:

$$b_{z_0}(z) := \frac{\overline{z_0}}{|z_0|}\frac{z - z_0}{z\overline{z_0} - 1},$$

- φ_{out} is non-zero and analytic: $\varphi_{\text{out}} = \exp(\text{rational})$.

Uniqueness of Null LQDs

A domain Ω is referred to as a *null LQD* if $\Omega \in \mathrm{QD}_0(0)$. We prove the following classification theorem for null LQDs.

Theorem 4.9

A domain is a null LQD if and only if it is a disk or exterior disk centered at 0*.*

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Question: Why are disks null LQDs but not null classical QDs?

Answer: If $\Omega \in \mathrm{QD}_0(0)$, then Ω must contain either 0 or ∞ because otherwise $1 \in L^1_a(\Omega; \rho_0)$, so

$$0 < \int_\Omega |w|^{-2} dA(w) = \oint_{\partial\Omega} 1 \cdot 0 dw = 0.$$

Whereas, for classical QDs $\Omega \in \mathrm{QD}(0)$ implies only $\infty \in \Omega$.

The Faber Transform Method For LQDs

Theorem 4.10

Let $\Omega \in QD_0(h)$ be a bounded simply connected domain with Riemann map φ. Then there exists rational $r \in \mathcal{A}_0(\mathbb{D}_*)$ such that[4]

$$\begin{aligned} \varphi(z) &= w_0 e^{r^\#(z)}, && \text{when } 0 \notin \Omega, \\ \varphi(z) &= \frac{w_0}{|z_0|} b_{z_0}(z) e^{r^\#(z)}, && \text{when } 0 \in \Omega, \end{aligned} \tag{2}$$

Where $\varphi(0) = w_0$ and $\varphi(z_0) = 0$.

[4] $r^\#(z) := \overline{r(\overline{z^{-1}})}$.

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Where $\varphi(0) = w_0$ and $\varphi(z_0) = 0$. In this case,

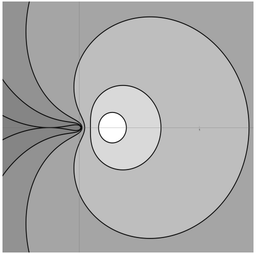
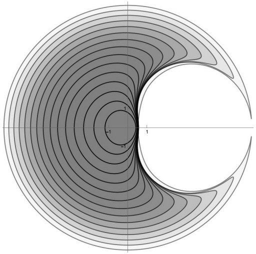
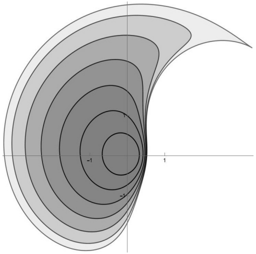
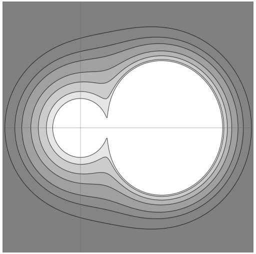
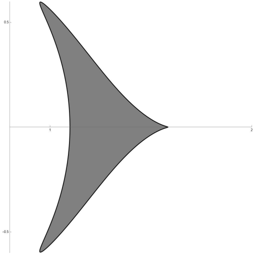
$$r(z) = \Phi_\varphi^{-1}\left(wh(w) + \operatorname*{Res}_{\infty}(h)\right)(z), \tag{3}$$

and

$$h(w) = \frac{\Phi_\varphi(r)(w) + C}{w} \tag{4}$$

[4] $r^{\#}(z) := \overline{r(\overline{z^{-1}})}$.

Classification of One Point LQDs.

	Ω **Bounded**	Ω **Unbounded**
$0 \notin \Omega$		 
$0 \in \Omega$		

Non-Singular Bounded One Point LQDs

Theorem 4.11

Fix $\alpha > 0$ and $w_0 \in \mathbb{C} \setminus \{0\}$. If $0 \notin \Omega$ is simply connected and bounded, then $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ iff $0 < \alpha \leq \pi^2$ and $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$.

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$\Longrightarrow$ **"Proof":** Suppose $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w - w_0}\right)$ with Riemann map φ s.t. $\varphi(0) = w_0$. By the theorem,

$$\varphi(z) = w_0 e^{r^{\#}(z)},$$

where

$$r(z) = \alpha w_0 \Phi_\varphi^{-1}\left(\frac{1}{w - w_0}\right)(z) = \alpha w_0 \frac{\psi'(w_0)}{z - \psi(w_0)} = \frac{\alpha w_0}{\varphi'(0)} \frac{1}{z}$$

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Hence, $\varphi(z) = w_0 e^{\frac{\alpha w_0}{\varphi'(0)} z}$. Solving for $\varphi'(0)$, we recover

$$\varphi(z) = w_0 e^{z\sqrt{\alpha}},$$

which is univalent in $\mathbb{D}$ iff $0 < \alpha \leq \pi^2$.

Non-Singular Unbounded One Point LQDs.

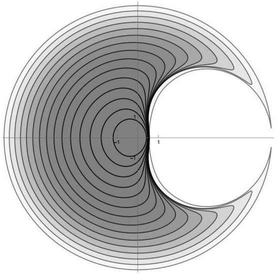
Theorem 4.12

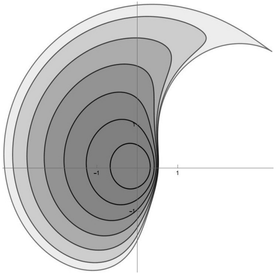
Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists an unbounded simply connected domain Ω not containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ if and only if there exist $c > 0$, $\lambda \in \mathbb{C}$, and $z_1 \in \mathbb{D}_*$ for which $\Omega = \varphi(\mathbb{D}_*)$, where φ is univalent and given by

$$\varphi(z) = cze^{\frac{\lambda}{1-z\overline{z_1}}},$$

where $\varphi(z_1) = w_0$, and $\lambda = \frac{\overline{\alpha w_0}}{\overline{z_1 \varphi'(z_1)}}$.

$w_0 = 1.9$
$\alpha = 1.5, 1.5 + .3i$





Singular Bounded One Point LQDs.

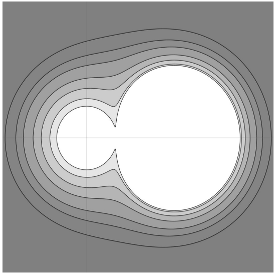
Theorem 4.13

Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists a bounded simply connected domain Ω containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ if and only if there exist $\lambda \in \mathbb{C}$ and $z_0 \in \mathbb{D}$ for which $\Omega = \varphi(\mathbb{D})$, where φ is univalent and given by

$$\varphi(z) = \frac{w_0}{|z_0|} b_{z_0}(z) e^{\lambda z},$$

where $\lambda = \frac{\overline{\alpha}}{\varphi'(0)}$.

$w_0 = 1$
$\alpha = .7$



Singular Unbounded One Point LQDs.

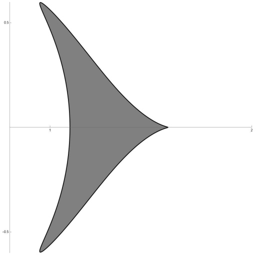
Theorem 4.14

Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists an unbounded simply connected domain Ω containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ if and only if there exist $\lambda \in \mathbb{C}$ and $z_0, z_1 \in \mathbb{D}_*$, for which $\Omega = \varphi(\mathbb{D}_*)$, where φ is univalent and given by

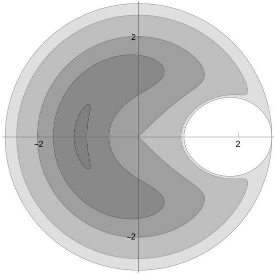
$$\varphi(z) = c|z_0| z b_{z_0}(z) e^{\frac{\lambda}{1-z\overline{z_1}}},$$

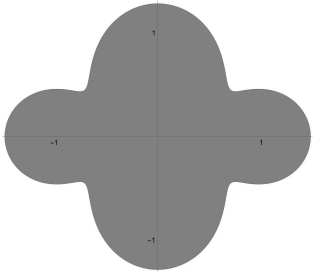
where $\varphi(z_1) = w_0$ and $\lambda = \frac{\overline{\alpha w_0}}{\overline{z_1}\,\overline{\varphi'(z_1)}}$.

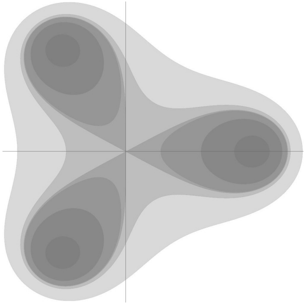
$w_0 = 2,$
$\alpha = -.15$



Chapter VI: Algebraic Quadrature Domains







Weighted QDs Summary

Class	Weight	Schwarz Boundary Values	Riemann map	Singularities
QDs	1	$S(w) \doteq \overline{w}$	φ rational	∞
PQDs	$\lvert w\rvert^{2(a-1)}$	$S_a(w) \doteq \frac{1}{a}\overline{w}\lvert w\rvert^{2(a-1)}$	φ_{out}^a rational	∞ 0
LQDs	$\lvert w\rvert^{-2}$	$S_0(w) \doteq \frac{\ln\lvert w\rvert^2}{w}$	$\ln \circ \varphi_{\text{out}}$ rational	$\infty, 0$
AQDs	$\lvert R'(w)\rvert^2$	$S_{R,\Omega}(w) \doteq R'(w)\overline{R(w)}$	$R \circ \varphi$ rational	Poles of R' Zeros of R'

Algebraic Quadrature Domains

Definition 5.1 (Algebraic quadrature domain)

Let $\Omega \subset \widehat{\mathbb{C}}$ be a rectifiable domain and R a non-constant rational function with $R \in C^0(\partial\Omega)$. We say that Ω is an *algebraic quadrature domain with respect to* R if there exists a rational function h such that

$$\int_\Omega f|R'|^2 dA = \oint_{\partial\Omega} f(w)h(w)dw, \qquad \forall f \in \mathcal{F}_R(\Omega).$$

This is denoted by $\Omega \in \mathrm{QD}_R(h)$.

That is, AQDs are quadrature domains with respect to the weight:

$$\rho_R(w) = |R'|^2.$$

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  1. The weight ρ_R may have multiple singularities.
  2. The weight is not necessarily radially symmetric.

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- Two primary features distinguish AQDs from other classes of QDs:
  1. The weight ρ_R may have multiple singularities.
  2. The weight is not necessarily radially symmetric.
- We refer to AQDs containing a root or pole of R' as *singular*.

Null AQDs.

We refer to a domain Ω as a *null* AQD if $\Omega \in \mathrm{QD}_R(0)$.

Theorem 5.2 (Partial classification of null AQDs)

If Ω is a domain containing no roots of R', then $\Omega \in QD_R(0)$ if and only if Ω is a connected component of $\{w \in \widehat{\mathbb{C}} : |R(w)| > a\}$ for some $a > 0$.

Even when Ω contains a root of R', Ω remains a null AQD for all cases checked. This suggests the following (unproven) complete classification result for null AQDs:

$$\Omega \in QD_R(0) \iff \exists\ a > 0 \text{ s.t. } \Omega \text{ is a connected component of } \{w : |R(w)| > a\}.$$

Theorem 5.3 (Change of variables for AQDs)

Let $\Omega, \Omega_R \subset \widehat{\mathbb{C}}$ be domains and suppose $R : Cl(\Omega_R) \to Cl(\Omega)$ is a bijective rational map with $R \in C^0(\partial\Omega_R)$. Then $\Omega \in QD(h)$ if and only if $\Omega_R \in QD_R(h_R)$, where h and h_R are related by

$$h_R(w) = \mathcal{C}_{(\Omega_R)_*}\left[h \circ R \cdot R'\right](w) = \oint_{\partial\Omega_*} \frac{h(\xi)d\xi}{R^{-1}(\xi) - w},$$

$$h(w) = \mathcal{C}_{\Omega_*}\left[\left(\frac{h_R}{R'}\right) \circ R^{-1}\right](w) = \oint_{\partial(\Omega_R)_*} \frac{h_R(\xi)d\xi}{R(\xi) - w}.$$

For example, if $\Omega_R \in \mathrm{QD}_R\left(\frac{\alpha}{w-w_0}\right)$, and R is injective on $\mathrm{Cl}(\Omega_R)$, then $R(\Omega) \in \mathrm{QD}(h)$, where

$$h(w) = \oint_{\partial(\Omega_R)_*} \frac{\alpha}{\xi - w_0} \frac{d\xi}{R(\xi) - w} = \frac{\alpha}{w - R(w_0)}.$$

Summary of Equivalent Characterizations

The following are equivalent to $\Omega \in \mathrm{QD}_R(h)$:

1. **Electrostatics:** $U(w) = h(w)$ in Ω_*, where U is the (appropriately renormalized) ρ_R-weighted Cauchy transform of Ω.

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2. **Generalized coincidence equation:**

$$R'(w)\overline{R(w)} = h(w) + \widetilde{G}(w), \quad \text{on} \quad \partial\Omega,$$

for some $\widetilde{G}$ analytic in Ω, except for simple poles at the poles of R in Ω.

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4. **The Riemann map:** If φ is the Riemann map associated to Ω, then $R \circ \varphi$ is rational.

Faber transform method for AQDs.

Theorem 5.4 (Inverse problem for bounded AQDs)

Suppose $\Omega \in QD_R(h)$ is a bounded simply connected domain with Riemann map $\varphi : \mathbb{D} \to \Omega$, where $\varphi(0) = w_0$ is not a pole of R. In this case,

$$R \circ \varphi(z) = R(w_0) + \Phi_\varphi^{-1}\left(\mathcal{C}_{\Omega_*}[R]\right)(z) - \Phi_\varphi^{-1}\left(\mathcal{C}_{\Omega_*}[R]\right)(0) + r^{\#}(z),$$

where

$$r(z) = \Phi_\varphi^{-1}\left(\mathcal{C}_{\Omega_*}\left[\frac{h(w) + G(w) + \sum_{j=1}^{m} \frac{q_j}{w - p_j}}{R'(w)}\right]\right)(z).$$

In particular, $R \circ \varphi$ is a rational function depending only on R, h, and w_0.

Faber transform method for AQDs.

Theorem 5.5 (Inverse problem for unbounded AQDs)

Suppose $\Omega \in QD_R(h)$ is an unbounded simply connected domain with Riemann map $\varphi : \mathbb{D}_* \to \Omega$, where $\varphi(\infty) = \infty$ and $\varphi'(\infty) = c > 0$. In this case,

$$(R \circ \varphi)(z) = \Phi_\varphi^{-1}\left(\mathcal{C}_{\Omega_*}[R]\right)(z) + r^\#(z) - r^\#(0),$$

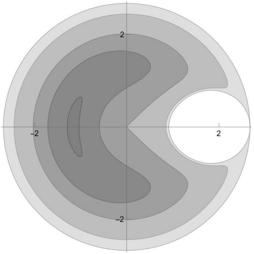
where

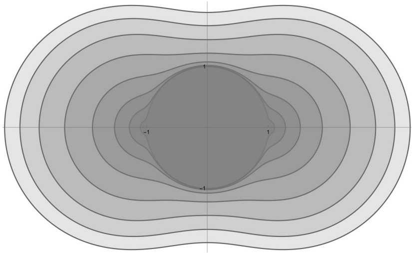
$$r(z) = \Phi_\varphi^{-1}\left(\mathcal{C}_{\Omega_*}\left[\frac{h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}}{R'(w)}\right]\right)(z).$$

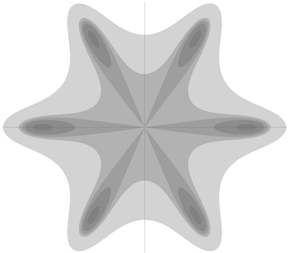
In particular, $R \circ \varphi$ is a rational function depending only on R, h, and c.

Summary

- Developed a Faber-transform framework for weighted quadrature domains.
- Obtained classifications of several families classical, power-weighted, log-weighted, and algebraic QDs.
- Identified new singular boundary behavior arising from singular weights.
- Identified connections between weighted QDs, the Riemann map, the Schwarz function, electrostatics, and logarithmic potential theory.







That's all!

Thank you!

Appendix

The Inverse and Direct Problems for LQDs

Question: When is an LQD uniquely associated to its quadrature function? How can we recover it?

These are open problems even for *classical* QDs.

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- $\Omega \in QD(0)$ (Ω is a "null" QD) iff Ω is an exterior disk
- If $\Omega_1, \Omega_2 \in QD(h)$ are star-shaped wrt a common point, then $\Omega_1 = \Omega_2$.[5]

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Some known results for QDs:

- The disk is the unique finitely connected *bounded* domain with the MVP.
- $\Omega \in QD(0)$ (Ω is a "null" QD) iff Ω is an exterior disk
- If $\Omega_1, \Omega_2 \in QD(h)$ are star-shaped wrt a common point, then $\Omega_1 = \Omega_2$.[5]
- If a s.c. bounded domain $\Omega \in QD(h)$ for some $h(w) = \sum_{j=1}^{n} \frac{c_j}{w - w_j}$ with the $c_j > 0$ and $\operatorname{diam}(\{w_j\}) < \sqrt{c_1 + \ldots + c_n}$, then Ω is unique.[6]

[5] Novikoff, P.S.: Sur le problème inverse du potentiel. (1938)

[6] Gustafsson, B., Putinar, M.: Selected topics on quadrature domains. (2007)

[7] Aharonov, D., Shapiro, H.S.: Domains on which analytic functions satisfy quadrature identities. (1976)

The Inverse and Direct Problems for LQDs

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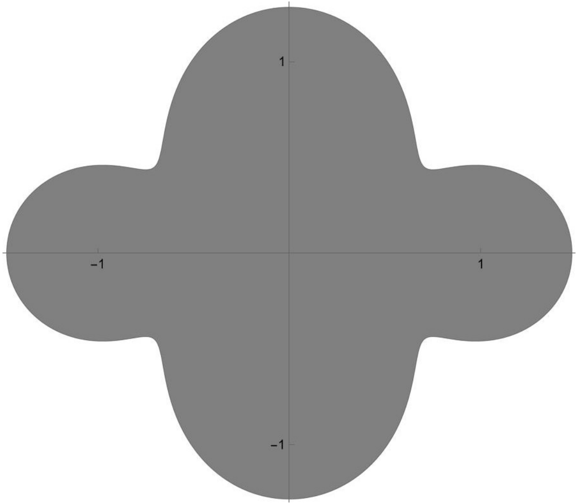
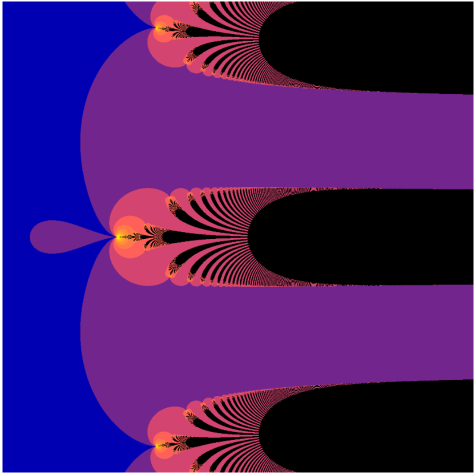
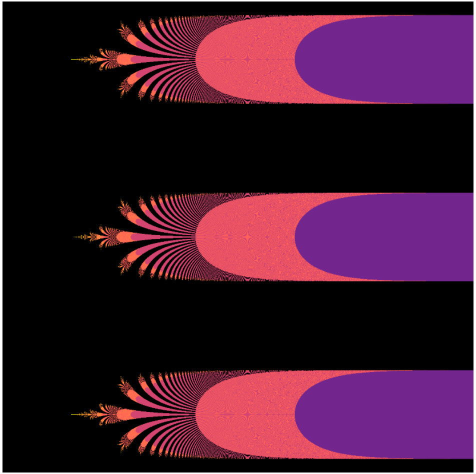
AQD Example



$$\Omega \in \mathrm{QD}_R\left(\frac{2}{3w}\right), \text{ where } R(w) = \frac{1}{w^2-1}.$$





Tiling set of the Schwarz reflection σ associated to $\Omega \in \mathrm{QD}_0(1)$ (left) and filled Julia set of the anti-holomorphic exponential map $w \mapsto e^{\overline{w}-1}$ (right).

PQD Power Map Relations

Lemma 7.1

Fix $a > 0$, $\alpha \in \mathbb{C} \setminus \{0\}$, *and* $k \in \mathbb{Z}_+$. *If* $\Omega \in QD_{\frac{a}{k}}(\alpha k^2)$ *then* $\{w : w^k \in \Omega\} \in QD_a\left(\alpha k w^{k-1}\right)$.

Lemma 7.2 (Converse to Lemma 7.1)

Fix $a > 0$, $\alpha \in \mathbb{C} \setminus \{0\}$, *and* $k \in \mathbb{Z}_+$. *If* $\Omega \in QD_a(\alpha k w^{k-1})$ *and is* $\mathbb{Z}_k$*−rotationally symmetric* ($e^{\frac{2\pi i}{k}}\Omega = \Omega$), *then* $\Omega^k \in QD_{\frac{a}{k}}(\alpha k^2)$.

One-point QDs: Existence proof

1. Suppose that $c \in \mathbb{C} \setminus \mathbb{R}$ is such that $Q(w) = |w|^2 - 2\mathrm{Re}(c \ln(w-2))$ has a local minimum $z_0 \neq 2$.

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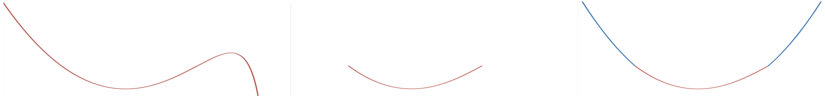

singular potential *localized potential* *modified potential*

Caltech

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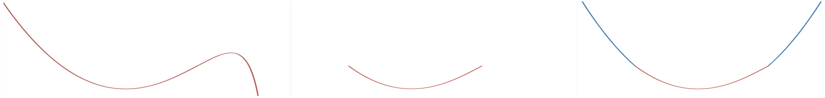

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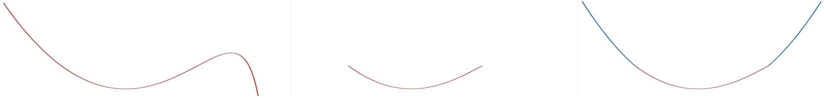

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4. So if Q has a local minimum $z_0 \neq 2$, then $\exists \Omega_t \in \mathrm{QD}\left(\frac{c}{w-2}\right)$ for some $t > 0$.

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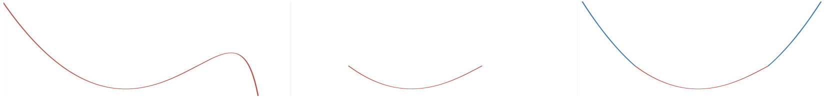

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Calculus exercise: Show that $Q(w) = |w|^2 - 2\text{Re}(c \ln(w-2))$ has a local minimum $z_0 \neq 2$ iff $2 + \text{Re}(c) - |c| > 0$ and the minimum is unique when it exists.

Proof of Uniqueness

Let $\Omega_t, \Omega_t' \in \mathrm{QD}\left(\frac{c}{w-2}\right)$ with $0 < t = \mu(\Omega_t) = \mu(\Omega_t')$.

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4. Finally, by localizing to an open nbhd of $K_t \cup K_t'$ and applying Frostman's theorem, we find that $K_t = K_t'$, so $\Omega_t = \Omega_t'$.

Cusp Formation

Recall that if $\Omega \in \mathrm{QD}(h)$ is s.c. and unbounded then there exists $a > 0$ such that $\varphi(z) = az + \Phi_{\varphi}^{-1}(h)^{\#}(z)$.

Thus when $h(w) = \frac{c}{w-2}$ there exists $a > 0$ such that

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Applying the additional Faber transform relation, $h = \Phi_{\varphi}\left(\varphi^{\#}\right)$, we obtain

$$\frac{c}{w-2} = h(w) = \Phi_{\varphi}\left(\varphi^{\#}\right)(w) = \frac{1}{w-2}\frac{(az_0|z_0|^2 - 2)(a\overline{z_0} - 2)}{|z_0|^2}.$$

Proof of Cusp Formation

$$c = (az_0|z_0|^2 - 2)(a\overline{z_0} - 2)|z_0|^{-2}, \qquad \overline{c} = (a\overline{z_0}|z_0|^2 - 2)(az_0 - 2)|z_0|^{-2}$$

Considering the obtained equation along with its complex conjugate and eliminating $\overline{z_0}$, we obtain a sextic in z_0,

$$0 = z_0^6 + z_0^5\left(\frac{a}{2} + O(1)\right) + z_0^4 O(1) + z_0^3 O\left(a^{-1}\right) + z_0^2 O(1) + z_0 O\left(a^{-1}\right) + 4a^{-2}.$$

[8] Y. Ameur, N-G. Kang, N. Makarov., "Scaling limits of random normal matrix processes at singular boundary points". (2020)

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The former is impossible because $|z_0| > 1$, and the latter is because then

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Thus $\nexists \Omega_t$ for t sufficiently large, so there exists a maximal t. By Sakai's theorem[8], $\partial\Omega_{t_*}$ must contain a $(\nu, 2)-$cusp, where $\nu \in 3 + 4\mathbb{N}_0$. That $\nu = 3$ follows from analysis of φ_{t_*}.

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[8] Y. Ameur, N-G. Kang, N. Makarov., "Scaling limits of random normal matrix processes at singular boundary points". (2020)

Characterizations of AQDs I: Electrostatics

Theorem 7.3

Fix a rational function R and a domain Ω whose boundary contains no poles of R, and $\{p_j\}_{j=1}^m$ are the poles of R in Ω. Then $\Omega \in QD_R(h)$ if and only if there exist positive constants $\{r_j\}_{j=1}^m$ such that

$$C_{\rho_R}^{\Omega\setminus U_+}(w) - \sum_{j=1}^{m} \frac{r_j^2}{w - p_j} = h(w) + \sum_{j=1}^{m} \frac{q_j}{w - p_j}$$

for all $w \in \Omega_*$.

$\overline{C_{\rho_R}^{\Omega\setminus U_+}(w)} - \overline{\sum_{j=1}^m \frac{r_j^2}{w-p_j}}$ corresponds to the electrostatic field due to a (renormalized) charge density $\rho_R = |R'|^2$ on Ω.

Characterizations of AQDs II: The Coincidence Equation

Theorem 7.4

If $\Omega \in QD_R(h)$ and $\{p_j\}_{j=1}^m$ are the finite poles of R in Ω, then there exists $G \in \mathcal{A}(\Omega)$ and constants $q_j \in \mathbb{C}$ such that

$$R'(w)\overline{R(w)} = h(w) + G(w) + \sum_{j=1}^{m} \frac{q_j}{w - p_j}, \quad \text{on} \quad \partial\Omega$$

Theorem 7.5

If R is a rational function and Ω is a domain with $\partial\Omega$ containing no poles of R, then $\Omega \in QD_R$ if and only if it admits a generalized Schwarz function $S_{R,\Omega}$. In this case,

$$S_{R,\Omega} = h(w) + G(w) + \sum_{j=1}^{m} \frac{q_j}{w - p_j},$$

where $G \in \mathcal{A}(\Omega)$, $\Omega \in QD_R(h)$, the p_j are the poles of R in Ω.

Characterizations of AQDs III: The Generalized S-Function

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where $G \in \mathcal{A}(\Omega)$, $\Omega \in QD_R(h)$, the p_j are the poles of R in Ω.

$S_{R,\Omega}$ is called a *generalized Schwarz function* for Ω if

1. $S_{R,\Omega} \in \mathcal{M}(\Omega)$,
2. $S_{R,\Omega}$ extends continuously to $\partial\Omega$,
3. $S_{R,\Omega} = R'\overline{R}$ on $\partial\Omega$.

Characterizations of AQDs IV: The Riemann Map

Let Ω be a simply connected domain with Riemann map φ. Then,

Theorem 7.6

$\Omega \in QD_R$ if and only if $R \circ \varphi$ extends to a rational function.

This is key to relating the Riemann map and quadrature function of AQDs.

Summary of Equivalent Characterizations of AQDs

The following are equivalent to $\Omega \in \mathrm{QD}_R(h)$:

1. The weighted Cauchy transform:

$$C_{\rho_R}^{\Omega \setminus U_+}(w) - \sum_{j=1}^{m} \frac{r_j^2}{w - p_j} = h(w) + \sum_{j=1}^{m} \frac{q_j}{w - p_j}$$

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2. Ω satisfies the generalized coincidence equation:

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4. $R \circ \varphi$ extends to a rational function (if simply connected).

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- Generalize beyond simply connected domains.

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- Analyze case in which $\partial\Omega$ contains a singular point.

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  - Classical case: Lee & Makarov (2016)

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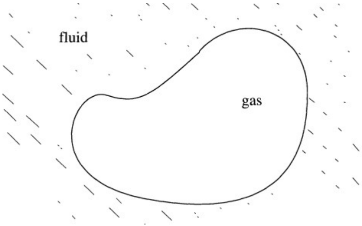

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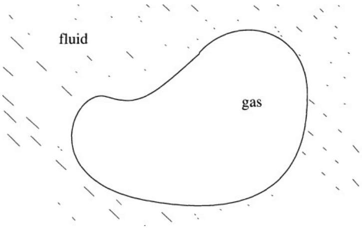

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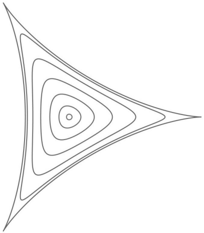

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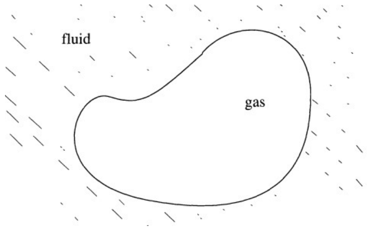

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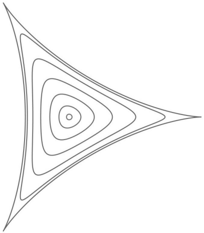

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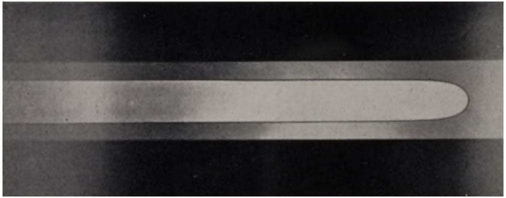
Remark: $\Omega_{t_0} \in \mathrm{QD}(h) \implies \Omega_{t_0+\delta t} \in \mathrm{QD}\left(h(w) + \frac{\delta t}{w - w_0}\right)$, where w_0 is the injection point.

Saffman-Taylor fingers



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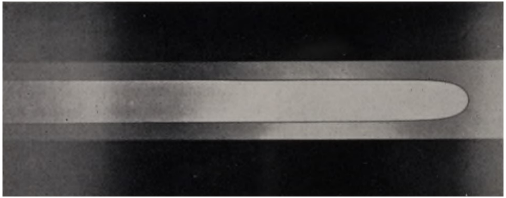

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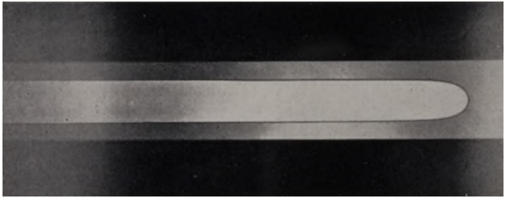

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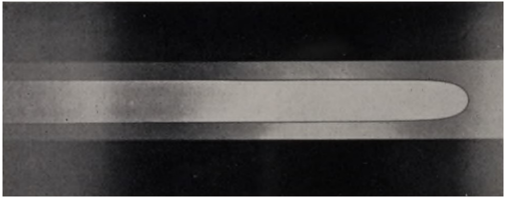

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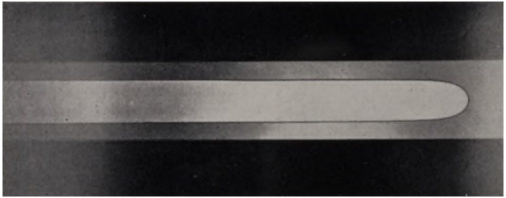

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Problem: ∞ in boundary

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We call a bounded domain $\Omega \subset \mathbb{C}$ a weighted *Abelian quadrature domain* wrt the weights $\rho : \Omega \to \mathbb{R}_{\geq 0}$ and $\Lambda \in \mathrm{Rat}(\Omega^c)$ if $\exists h = r + \Lambda L$ for $r, e^L \in \mathrm{Rat}(\Omega)$, such that

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Unbounded case is similar.

Weighted Abelian QD Example

If $0 \notin \Omega \in \widetilde{\mathrm{QD}}_{|w|^{-2}}(h)$ is s.c, can still obtain a Faber transform formula

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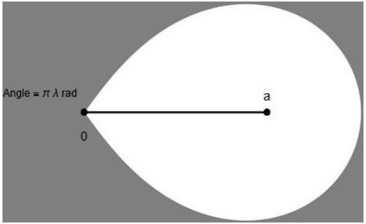

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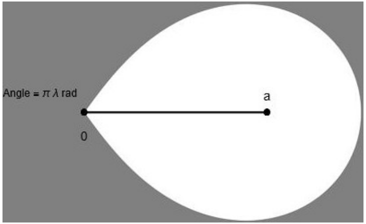

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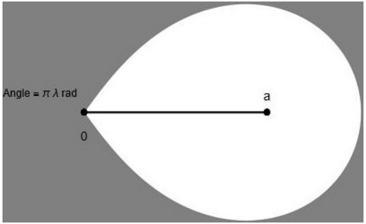

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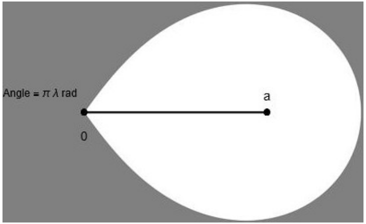

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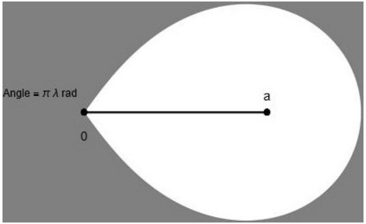

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So if $f \in L^1_a(a\mathbb{D}(1)^\lambda; |w|^{-2})$,

$$\frac{1}{\pi}\int_{a\mathbb{D}(1)^\lambda} \frac{f(w)}{|w|^2}\,dA(w) = \lambda\int_0^a \frac{f(w)}{w}\,dw$$

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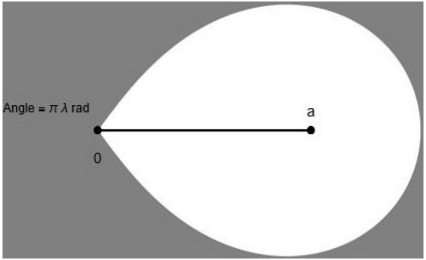
Weighted Abelian QD Example (cont.)

Changing variables $w \mapsto \ln(w)$: (Saffman-Taylor finger)



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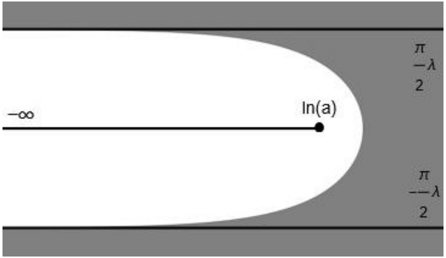

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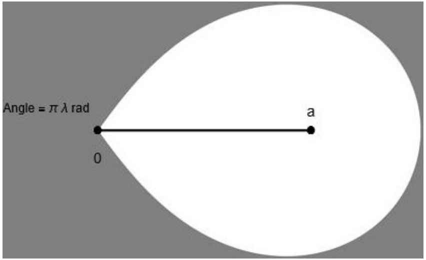

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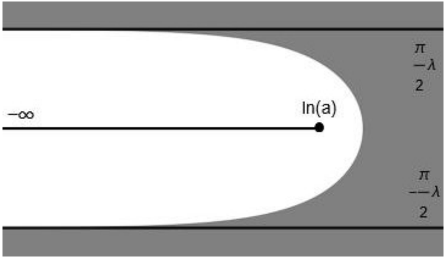

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Gives

$$\frac{1}{\pi}\int_{\ln(a)+\lambda\ln(\mathbb{D})} f dA = \lambda \int_{-\infty}^{\ln(a)} f(w)dw$$

For $f \in L^1_a(\ln(a) + \lambda \ln(\mathbb{D}(1)))$